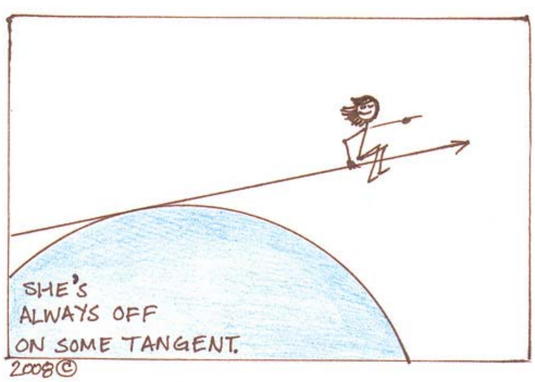
Advance
KENTUCKY

AP Calculus Study Session Handout

Approximating Derivatives and Tangent Line Approximations



Approximating Derivatives and Local Linearity

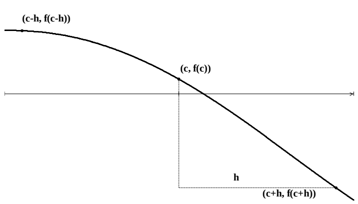
Approximating Derivatives

On the AP Calculus Exams, students are often asked to approximate derivatives given function values in a table. In these problems, students are expected to approximate the slope of the tangent line at the indicated points using the slope of a secant line that is close to that point. Students should select points for their approximation that are as close as possible to the desired point of tangency. Often the point of tangency is not included in the given table.

Difference Quotients

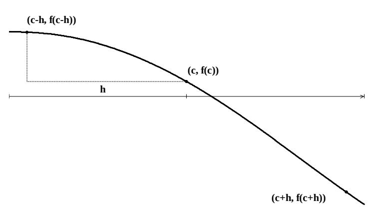
Three ways to approximate the slope of the tangent are to use forward, backward, and symmetric difference quotients.

Forward Difference Quotient



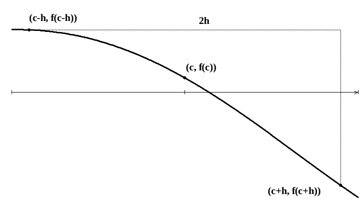
$$f'(c) \approx \frac{f(c+h) - f(c)}{h}$$

Backward Difference Quotient



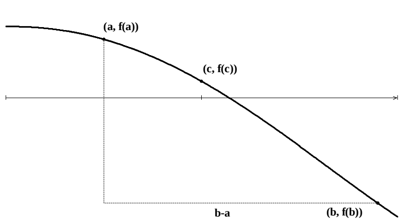
$$f'(c) \approx \frac{f(c) - f(c-h)}{h}$$

Symmetric Difference Quotient



$$f'(c) \approx \frac{f(c+h) - f(c)}{h}$$

If you aren't given points that are symmetric about the point of tangency, use the closest points on either side of the point of tangency to approximate the derivative.



$$f'(c) \approx \frac{f(b) - f(a)}{b - a}$$

2008 AB 2 BC 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon ($t = 0$) and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 p.m. ($\boldsymbol{t = 5.5}$). Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 p.m. ($t = 3$), to the nearest whole number?

2005 AB 3 BC 3

Distance x cm	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

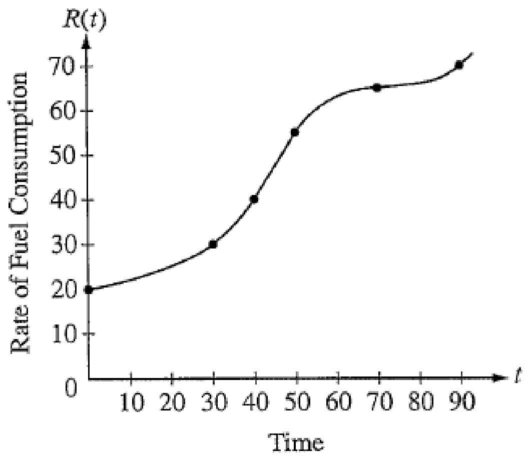
(a) Estimate $\boldsymbol{T'(7)}$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

2003 AB 3



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function *R* of time *t*. The graph of *R* and a table of selected values of *R*(*t*), for the time interval $0 \le t \le 90$ minutes, are shown above.

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time *t* = 45 minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

2001 AB 2 and BC 2

t (days)	$W(t)$ $(°C)$
0	20
3	31
6	28
9	24
12	22
15	21

The temperature, in degrees Celsius $(°C)$, of the water in a pond is a differentiable function *W* of time *t*. The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(a) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

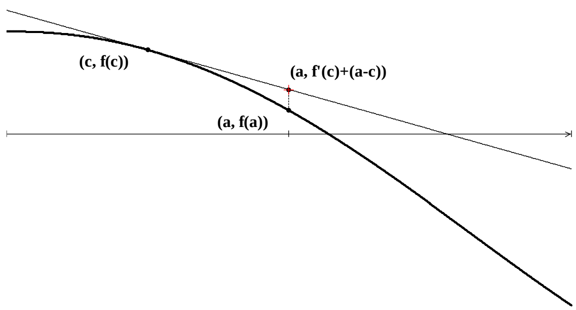
(b) A student proposes the function *P*, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time *t*, where *t* is measured in days and *P*(*t*) is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(c) Use the function *P* defined in part (c) to find the average value, in degrees Celsius, of *P*(*t*) over the time interval $0 \le t \le 15$ days.

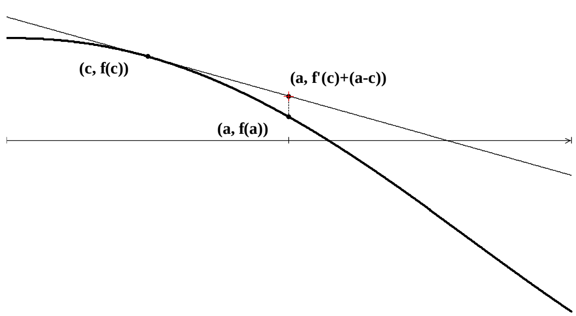
Tangent Line Approximations

On the AP Calculus Exams, students are expected to be able to approximate function values using a tangent line.

$$f(a) \approx f'(c) + (a - c)$$

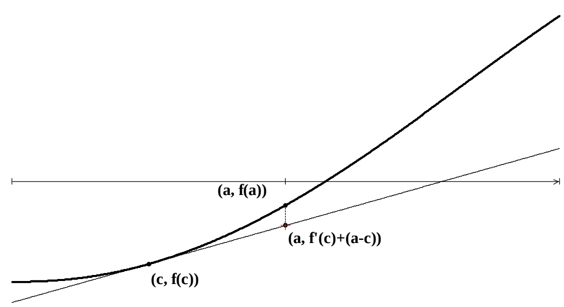


Students are also expected to know if a tangent line approximation is greater than or less than the actual function value. This can be determined by the concavity of the original function.



Graph of ***f*** **is concave down** on the interval containing the point of tangency, the **tangent line lies above** the curve.

The tangent line approximation is **greater than** the actual value.



Graph of ***f*** **is concave up** on the interval containing the point of tangency, the tangent line lies below the curve.

The tangent line approximation is **less than** the actual value.

2007 AB5 BC5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.3	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \leq t \leq 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $\boldsymbol{t = 5.4}$ using the tangent line approximation at $\boldsymbol{t = 5}$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

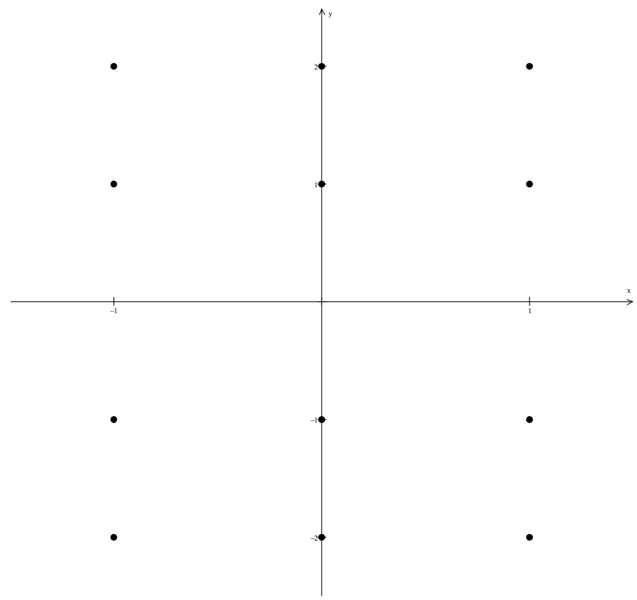
(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)dt$? Give a reason for your answer.

2005 AB 6

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



(b) Let $\boldsymbol{y = f(x)}$ be the particular solution to the differential equation with the initial condition $\boldsymbol{f(1) = -1}$. Write an equation for the line tangent to the graph of $\boldsymbol{f}$ at (1, -1) and use it to approximate $\boldsymbol{f(1.1)}$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

2002 AB 6

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let *f* be a function that is differentiable for all real numbers. The table above gives the values of *f* and its derivative f' for selected points *x* in the closed interval $-1.5 \le x \le 1.5$. The second derivative of *f* has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(a) Evaluate $\int_0^{1.5} \left(3f'(x) + 4\right) dx$. Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

(c) Find a positive real number *r* having the property that there must exist a value *c* with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

(d) Let *g* be the function given by $g(x) = \begin{cases} 2x^2 - x - 7 & \text{for } x < 0 \\ 2x^2 + x - 7 & \text{for } x \ge 0. \end{cases}$ The graph of *g* passes through each of the points $(x, f(x))$ given in the table above. Is it possible that *f* and *g* are the same function? Give a reason for your answer.

Name: ________________________ **Class:** ___________________ **Date:** __________ **ID: A**

Local Linear Approximation

Multiple Choice
Identify the choice that best completes the statement or answers the question.

____ 1.

x	0	0.5	1.0	1.5	2.0
$f(x)$	3	3	5	8	13

A table of values for a continuous function f is shown above. If four equal subintervals of [0, 2] are used, which of the following is the trapezoidal approximation of $\int_0^2 f(x)\,dx$?

a. 8
b. 12
c. 16
d. 24
e. 32

____ 2. Let f be the function given by $f(x) = x^2 - 2x + 3$ The tangent line to the graph of f at $x = 2$ is used to approximate values of $f(x)$. Which of the following is the greatest value of x for which the error resulting from this tangent line approximation is less than 0.5 ?

a. 2.4
b. 2.5
c. 2.6
d. 2.7
e. 2.8